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LETTER TO THE EDITOR

Conduction of small constrictions in a magnetic field in the ballistic regime

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Abstract. The conductance of a constriction in a magnetic field in the ballistic regime is considered. It is shown that if the edges of the construction are smooth the conductance is quantised in units $2e^2/h$, which are independent of the magnetic field. The linear response theory and the quasi-classical approximation are used for the calculations.

In recent experiments on microscopic constructions a step-like dependence of the conductance on the width of the constriction was discovered (van Wees *et al* 1988a, Wharam *et al* 1988). The quantisation of the conduction in integer units of $2e^2/h$ is closely related to the possibility of electrons moving through the constriction in the ballistic regime. The first explanation of this effect was given by van Wees *et al* (1988a) and Wharam *et al* (1988) with the help of a one-dimensional model and the Landauer relations (Imry 1986). In this model each band corresponding to different wavevectors of transversal quantisation gives the contribution $2e^2/h$. It was also discovered that the quantisation of the conductance is not destroyed by a magnetic field (van Wees *et al* 1988b).

A one-dimensional model including a magnetic field and finite frequency was considered by Kramer and Mašek (1988) and Mašek and Kramer (1988, 1989), where Kubo-type formulae were used for calculations of the conductance. However, the constrictions available experimentally are far from being one dimensional. Besides Kramer and Mašek (1988) made an additional assumption that the electric field does not depend on the coordinates. But this assumption can in principle contradict the continuity equation.

A model of a constriction with slowly varying width was considered by Glazman *et al* (1988) where the shape of the steps was related to the geometry of the constriction. In this work all results were obtained in the absence of a magnetic field on the basis of the Landauer formula (Imry 1986). A model analogous to the one used in Glazman *et al* (1988) is considered below. Now the consideration includes the case of a non-zero magnetic field. All calculations are based on the Kubo formulae for a linear response, which also enable finite frequencies to be considered. The dependence of the electric field on the coordinate along the constriction is supposed to be arbitrary. In principle this dependence can be found after solving the Maxwell equations. However, as may be seen from the calculations presented below, in the regime of the ballistic transport at zero frequency the current through the constriction is completely determined by the

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voltage difference. In this case the information about the dependence of the electric field on the coordinate is not important.

The current can be written as the response to an electric field $\mathscr{E}(r)$ in the following form

$$j_\omega(r) = \int X_\omega(r, r')\mathscr{E}_\omega(r')\,\mathrm{d}r'\,\mathrm{d}r_\perp \tag{1}$$

where $r_\perp$ is the transversal coordinate. The real part $g_\omega(r, r')$ of $X_\omega(r, r')$ determines dissipation and has the following form

$$g_\omega(r, r') = \frac{2\pi}{\omega}\int (n(\varepsilon) - n(\varepsilon + \omega)) \sum_{\alpha,\beta} |J_{\alpha,\beta}|^2 \delta(\varepsilon - E_\alpha)\delta(\varepsilon + \omega - E_\beta) \tag{2}$$

where $n(\varepsilon)$ is the Fermi distribution, and $J_{\alpha,\beta}$ are the matrix elements of the current operator

$$\hat{\boldsymbol{J}} = \frac{e}{m}\left(-\mathrm{i}\frac{\partial}{\partial \boldsymbol{r}} - \frac{e}{c}\boldsymbol{A}\right).$$

For a homogeneous magnetic field B the Schrödinger equation takes the form

$$-(1/2m)[(\partial/\partial x) - (\mathrm{i}e/c)By]^2\,\Psi - (1/2m)(\partial^2\Psi/\partial y^2) + u(x, y)\Psi = E\Psi. \tag{3}$$

In (3) $u(x, y)$ is a potential describing the walls of the constriction. Glazman *et al* (1988) used certain boundary conditions instead of the potential $u(x, y)$. Depending on the experimental situation, it is supposed that electrons can move only in the plane $\{x, y\}$, the axis x being directed along the constriction, y being the transversal coordinate.

For further calculations let us assume that $u(x, y)$ slowly varies in x. In this case one can use a quasi-classical approximation when considering the dependence of Ψ on x. The dependence of this function on y can be strong and must be studied exactly.

In order to take into account these properties of the solution let us represent the solution $\Psi(x, y)$ in the form

$$\Psi(x, y) = \mathrm{e}^{\mathrm{i}\sigma(x)}\varphi(x, y, \sigma'(x)). \tag{4}$$

The function $\varphi(x, y, \sigma'(x))$ in (4) is a solution of the following equation

$$-(1/2m)(\partial^2\varphi/\partial y^2)(x, y, \sigma'(x)) + (1/2m)[\sigma'(x) - (eB/c)\,y]^2\varphi(x, y, \sigma'(x))$$
$$+ u(x, y)\varphi(x, y, \sigma'(x)) = \varepsilon(x, \sigma'(x))\varphi(x, y, \sigma'(x)). \tag{5}$$

The variable x enters (5) only as a parameter. Substituting (4) into (3) and using (5) one can obtain an equation for σ

$$-(1/2m)(\partial^2\varphi/\partial x^2) + \varepsilon(x, \sigma'(x))\varphi$$
$$-(\mathrm{i}/2m)[\sigma''(x)\varphi + 2(\partial\varphi/\partial x)(\sigma'(x) - eBy/c)] = E\varphi. \tag{6}$$

Till now no approximations have been made. To go further, let us neglect the first term in (6) and assume that solutions of (5) form a discrete spectrum, thus providing the quantisation of the transversal motion in the constriction. Eigenfunctions φ_n are supposed to be normalised for arbitrary x:

$$\int \varphi_n^2(x, y, \sigma'(x))\, \mathrm{d}y = 1. \tag{7}$$

Using the identity

$$\frac{1}{m}\int \left(\sigma'(x) - \frac{eB}{c}y\right) \varphi_n^2(x, y, \sigma'(x))\, \mathrm{d}y = \frac{\partial \varepsilon(x, \sigma'(x))}{\partial \sigma'} \tag{8}$$

which follows immediately from (6) and (7), neglecting the term $(1/2m)(\partial^2\varphi/\partial x^2)$ in (6), multiplying this equation by φ and integrating over y one can obtain the following equation

$$\varepsilon_n(x, \sigma'(x)) - E = \tfrac{1}{2}\mathrm{i}(\partial/\partial x)\, \partial\varepsilon_n(x, \sigma'(x))/\partial\sigma' \tag{9}$$

where $\varepsilon_n(x, \sigma'(x))$ is the nth eigenenergy of (5).

If the magnetic field B is equal to zero one obtains from (5)

$$\varepsilon_n(x, \sigma'(x)) = \varepsilon_n^{(0)}(x) + (1/2m)\sigma'^2(x) \tag{10}$$

where $\varepsilon_n^{(0)}(x)$ is the nth eigenenergy of the equation

$$-(1/2m)(\partial^2\varphi/\partial y^2) + u(x, y)\varphi = \varepsilon_n^{(0)}(x)\varphi. \tag{11}$$

In this case, substituting (10) into (9) one recovers the corresponding formula of Glazman *et al* (1988).

The assumption about a slow dependence of the potential u on x enables us to solve (9) approximately. Representing the solution $\sigma(x)$ of (9) in the form

$$\sigma_n(x) = \sigma_{n0}(x) + \sigma_{n1}(x)$$

where $\sigma_{n0}(x)$ is the solution of the equation

$$\varepsilon_n(x, \sigma'_{n0}(x)) - E = 0$$

and considering the RHS of (9) as a perturbation, one can find for the function $\Psi_n(x, y)$

$$\Psi_n^{\pm}(x, y) = (p_n(x))^{1/2} \exp[\pm\mathrm{i}(\sigma_{n0}(x) - \sigma_{n0}(0))]\, \varphi_n(x, y, \sigma'_n(x)) \tag{12}$$

where

$$p_n(x) = \partial\varepsilon_n(x, \sigma'_{n0}(x))/\partial\sigma'_{n0}.$$

The constricted area connects two large areas where the potential $u(x, y,) = 0$ and the electric field $\mathscr{E} = 0$. Therefore one must integrate in (1) only over the area of the constriction. In the limit $\omega \to 0$ the sum over the eigenstates in (2) reduces to the sum over diagonal elements of the current operator $\hat{J}$. In the considered quasiclassical approximation the action of the current operator $\hat{J}$ on a wavefunction $\Psi^{\pm}$ reduces to the following expression

$$\hat{J}\Psi_n^{\pm} = (e/m)[\pm\sigma'_n(x) - (eB/c)y]\, \Psi_n^{\pm} \equiv \pm e p_n(x)\Psi_n^{\pm}(x). \tag{13}$$

Substituting (12) and (13) into (2) one can see that the function $p_n(x)$ does not enter the result. Summation over eigenstates in (2) reduces to the integration over the energy E and summation over different transverse modes. Of course, for each energy E one should take into account the contribution of both the functions Ψ^+ and Ψ^-. The main contribution when integrating over coordinates comes from products of the type $\Psi^+\Psi^-$ (if the length of the constriction exceeds the wavelength). The result of the integration over energies E in (2) depends on the quantity

$$\varepsilon_{n\max} = \max_{(x,\sigma'(x))}[\varepsilon_n(x, \sigma'(x))].$$

The case $\varepsilon > \varepsilon_{n\max}$ corresponds to the absence of classical turning points, when particles can move without a reflection (strictly speaking an exponentially small reflection is possible). In this case the integration over E for the nth modes gives in the limit $\omega \to 0$ unity. Due to the normalisation conditions for the functions φ_n, the integration over the transversal coordinate y also gives unity. If $\varepsilon < \varepsilon_{n\max}$ the contribution of the nth model is proportional to the probability of tunnelling through a wide barrier and is small. It is interesting to note that in the limit $\omega \to 0$ only the integral

$$V = \int E(r)\, \mathrm{d}r \tag{14}$$

enters the final result for arbitrary $E(r)$. Substituting (12) into (1) and (2) and performing integration one can obtain, finally, for the conductance G

$$G = (2e^2/h)n_{\max}(\varepsilon_F) \tag{15}$$

where $n_{\max}$ is the maximal integer at which the inequality $\varepsilon_F > \varepsilon_{n\max}$ is still valid. The slope of curves connecting two different steps is large if the probability of tunnelling through a barrier is small. For smooth potentials $u(x, y)$ the form of the effective barrier determined by $\varepsilon(x, \sigma'(x))$ (5) is also smooth and the probability of the tunnelling is small. Only in the case when $\varepsilon(x, \sigma'(x))$ is smooth is the quantisation good.

The accuracy of the quantisation increases when increasing the magnetic field. In order to see this effect one should use (10), which can be written also for $B \neq 0$ if $(\sigma'(x))$ is not very large. Now $\varepsilon_n^{(0)}(x) \equiv \varepsilon_n(x, 0)$, where $\varepsilon_n(x, \sigma'(r))$ is the nth eigenenergy of (5), and one should substitute the electron mass m in (10) by an effective mass m^* depending on the field B. The energy $\varepsilon_n^{(0)}(x)$ becomes more smooth when the field B increases because the potential $u(x, y)$ is then less important. Besides, the effective mass m^* grows. Using (9) and (10) one obtains conventional quasi-classical formulae with the effective potential $\varepsilon_n^{(0)}(x)$, which becomes more smooth when increasing the magnetic field and with the growing mass m^*. Hence the quasi-classical approximation becomes better and tunnelling through a barrier more difficult. It results in a more perfect quantisation.

Glazman *et al* (1988) showed that the quantisation is good if $R \gg d$, where R is the radius of the curvature of walls and d is the width. This condition holds, provided also that the magnetic field is not very large. For large magnetic fields when the magnetic length γ becomes smaller than the condition of good quantisation becomes weaker $R \gg \gamma$. The quantisation for large n is worse than that one for $n \simeq 1$ because the potential $u(x, y)$ in (10) is in this case more important.

The increase of the accuracy of the quantisation when increasing the magnetic field can be seen from the experimental data presented by van Wees *et al* (1988b) and Wharam *et al* (1988).

At finite frequencies ω the current depends not only on the voltage difference but on the whole function $E(r)$. This function must be determined from the Maxwell equations. Formulae obtained by Kramer and Mašek (1988) for finite frequencies are based on the assumption that the field E is constant. It can give a good qualitative description of the transport but for a quantitative description one should take into account fields generated by non-uniform charge distributions.

In conclusion, it was shown that the conductance of a small constriction in a magnetic field is quantised if the shape of the constriction is smooth. Due to the electrostatic nature of the potential barriers forming the constrictions the assumption about a smooth shape of the potential determining the constriction can correspond to experiments. The magnetic field improves the quantisation.

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